ARITHMETICAL CONTINGENTISM

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There are, so we will be supposing in this paper, mathematical objects, such as sets, groups, and numbers.[1] A widely shared philosophical doctrine has it that these, unlike the concreta of our everyday experience, exist necessarily.[2] After all, whilst we can easily imagine things being such that Donald Trump or the planet Jupiter don't exist, it seems harder to contemplate the non-existence of the number two or the hereditarily finite sets. So, we are tempted to conclude, mathematical objects exist necessarily. There is, to invoke the familiar hereustic, no possible world in which there are no such objects.

The type of modality at issue here is of the broadest sort: metaphysical or broadly logical necessity. In actual fact, recent work by Chris Scrambler has rolled back on the consensus that all mathematical objects are necessary in this sense.[3] Drawing on potentialist treatments of set theory [7][8], he has invited us to treat the sense of possibility in which any given plurality of objects form a set as metaphysical [9]. Hence, in particular, the actual sets do not form a set, but possibly: there is a set whose elements are all and only the actual sets. Since this set might exist, but does not, it follows that not all mathematical objects exist of necessity.

A philosopher might accept so much but insist that sets bring with them peculiar philosophical difficulties; paradox ever lurks, and the fact that appeal to contingent existence is made in order to fend this off should in no way cause us to suppose that other, less problematic, mathematical objects exist contingently. In particular the *natural numbers* (hereafter *numbers*), where these are *sui generis* objects and not to be identified with set-theoretic implementations such as the von Neumann ordinals, are necessary existents, just as we always took them to be. My purpose here is to unsettle even this conviction by presenting an understanding of the numbers on which they exist contingently (although, in a clear sense which will become apparent,

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[1]If you're not prepared to accept this supposition, there is a nearby problem to the *de re* one about the necessary existence of mathematical objects, namely a *de dicto* one about the necessity of mathematical truth. See below.

[2]Those who, following van Inwagen, take God to be a concretum, might make an exception to the general view that concreta are contingent []. I reject this view, and this will be seen to be important in what follows.

[3]There's something of a nominalist parallel in [4]

they *nearly* exist of necessity. §1 below presents this understanding, and §2 defends it from an objection. The conclusion is that, to the extent that the understanding of number motivating the paper is attractive, we ought to withhold assent from the claim that numbers exist necessarily. Contingency, then, afflicts mathematical ontology way beyond the set-theoretic universe.[4]

1

Frege famously supported the view that numbers attach to concepts. One is the number attached to the concept *Current President of the United States*; three is the number attached to the concept *sister of the author of this paper*, and so on. Given a commitment to the necessary existence of concepts, such as that defended in [3], the necessity of numbers is quickly forthcoming. Take any possible world you like. In that world, there exists the concept *non-selfidentical*. Let zero be the number that numbers this. Now let one be the number that numbers the concept *zero*. And so on.

What, we might worry, if concepts are not necessary? What if, in Aristotelian fashion, we take concepts to depend on concrete objects for their existence? What, moreover, if there might have been no concreta; that is, what if what is sometimes called *metaphysical nihilism* is true [1]? Then things look very different – the world empty of concreta is also, by the dependence of concepts on concreta, empty of concepts (including the concept of non-selfidentiy) and the line of argument sketched above is blocked.

So there is at least metaphysical outlook on which the claim that numbers are contingent has a good deal of plausibility: the one that combines a Fregean approach to number, with Aristotelianism about concepts, and metaphysical nihilism. Here I want to explore an alternative outlook, one that modifies the Fregean approach to number. As a way into this I will endorse the Fregean platitude that central to our understanding of number has to be that we use numbers to *count*, but question Frege's denial of the ordinary language platitude that we use numbers to count *things* or objects. Of course, in the *Grundlagen* Frege has compelling arguments against the view that numbers attach to individual objects, as properties [2,]. But these do not touch the view that numbers attach to the objects they number, not as individuals, but as pluralities. I have elsewhere presented the view, as has Keith Hossack separately, that numbers are properties of pluralities, where a plurality just is some objects [][6]. Since there is no empty plurality, it follows, consonantly with the historical development of number systems, that zero is not a natural number. It will need to be recovered for mathematical

[4]Surely if natural numbers are contingent then the members of all other number systems are contingent. Suppose that the naturals are contingent. Now assume, without loss of generality, that the rationals exist at a natural-free world. Consider $\frac{1}{1}, \frac{2}{1}, \frac{3}{1} \ldots$ contradicting the supposition that the naturals don't exist at this world.

purposes in the treatment of the integers.

For all the conceptual shift here, much of the Fregean picture remains the same. So, taking one to be the first number and availing ourselves of ordered pairs (which can be introduced by abstraction [5]), it is straightforward to show that in standard plural logic an implementation of (second-order) Peano Arithmetic can be derived from:

(BP) $$\#xx = \#yy \leftrightarrow \exists zz\, BIJ(zz, xx, yy)$$

Where 'BIJ' abbreviates the statement in the language of plural logic that zz are the plural graph of a bijection from xx onto yy.

How do things look modally? Very different from the more standard Fregean case. In the nihilist world,[5] where no concreta exist, it is not true that $\exists x\, x = x$, since there is no witness for x (we are being thoroughly Aristotelian: any abstracta depend on concreta for their existence). Since appeal to this truth of classical logic[6] is indispensible in proving the plural equivalent of Frege's theorem, and – for that matter – in proving that a single number exists, we can have no assurance that any number exists at the nihilist world.

It is important to see that this is the right result. We ought not to be able to show that a number exists at the nihilst world since, on the plural-based understanding of numbers we are working with, no numbers do exist at that world. It is a conceptual truth that there is no empty plurality (a plurality is some things; there are no things such that nothing is one of those things!) Hence standard plural logics contain as an axiom,

(NON-EMPT) $$\forall xx \exists x\, x \prec xx$$

At the nihilist world, (NON-EMPT) is vacously true; there are no pluralities, because there are no objects to be members of pluralities. But numbers are properties of pluralities – the number one is the property a plurality has if it has a unique member – and we are assuming that properties do not exist uninstantiated. So there are no numbers at the nihilist world.

Since there is a world at which numbers do not exist, the existence of numbers is contingent. Note, however, that there is a clear manner in which the existence of numbers is *nearly* necessary. At any non-nihilist world the

[5] I leave aside from consideration whether worlds are individuated extensionally, and so whether there is a unique nihilist world.

[6] Note that if we want to keep our logic classical and also to be metaphysical nihilists, we need to abandon the claim that the truths of logic are necessary truths. They are, however, *nearly* necessary, holding at all but the nihilist world.

numbers exist, and if any number exists, all the numbers exist (each number other than one serving to number its predecessors, as in the standard Fregean construction).

The combination of views needed to get to this point – numbers as plural properties; Aristotelianism about properties, and metaphysical nihilism – are clearly on the table of contemporary metaphysics. In combination they constitute an outlook on which numbers exist contingently. That ought to suffice to undermine the idea that is *obvious*, or a datum not in need of defence, that numbers exist of necessity.

2

But this cannot be right, so runs a predictable line of objection: this outlook cannot reflect reality. Why? Because we can prove that many arithmetical truths hold;[7] and what is provable is necessary. Hence all normal modal logics contain,

(NEC) $$\vdash \phi \Rightarrow \vdash \Box\phi$$

The invocation of (NEC) here is too swift, however. In an arithmetical proof we do not have a case of $\vdash \phi$, but rather (effectively) one of $PA \vdash \phi$, and this does not permit necessitation. What we do have an absolute proof of, by the deduction theorem, is $\wedge PA \rightarrow \phi$, where $\ulcorner \wedge PA \urcorner$ designates the conjunction of the (plural) PA axioms.[8] So, by (NEC), $\Box(\wedge PA \rightarrow \phi)$. But the necessary truth of this conditional, carrying as it does no ontological commitments, is something the arithmetical contingentist can accept.

But isn't this missing the point, namely that mathematical truths are necessary truths? To see this, runs the new complaint, we do not need to appeal to modal logic. We need simply to reflect on our own grasp of mathematical concepts: don't they impose themselves upon us with a particular force? Isn't a consequence of this that we can't imagine mathematical truths not obtaining? But if we find ourselves compelled to accept that mathematical truths are necessary truths, then since some of these truths have existential import, we are compelled to accept the necessary existence of mathematical objects, and in particular of numbers, contrary to the outlook described above.

Whatever response the contingentist makes here, it ought to take on board what is undoubtedly the case, that mathematical truths compel us with a

[7] If not always *in arithmetic*. Consider e.g. the standard Godel sentence for PA. We'll ignore this kind of issue as beside present concern in what follows.

[8] Unlike the singular first-order case, this conjunction is finite, since plural quantification can be used to state induction as a single axiom.

particular force and that, at least regarding parts of mathematics such as elementary arithmetic, counter-mathematical thinking seems unthinkable.[9] It would be foolhardy to take a stand by challenging this terrain.

Instead the contingentist can seek to explain our intuitions in terms of her favoured outlook. Remember that the existence of numbers is nearly necessary according to that outlook. In particular all those worlds of the sort for which our cognitive faculties are well-equipped, worlds containing concreta which we can perceive and discuss, are worlds in which numbers exist. The nihilist world is an outlier, and easily forgotten when reasoning modally and otherwise. It's unsurprising that we take the seeming unthinkability of counter-arithmetical situations as signalling necessity, and they very nearly do, but there is an important exception. Our intuitions do not track the nihilist world.

We could leave matters there, with an affirmation that arithmetical truths are nearly necessary, and that this explains our intuitions about them. But there is the possibility of approaching even closer to philosophical orthodoxy. We can make a distinction between the claim about *de re* necessity, that numbers exist of necessity, and a claim about *de dicto* necessity, that arithmetical truths are necessary truths. The contingentist position developed above rules out the former claim (and the advice of the previous paragraph seems a good way to reconcile this with intuition), but might there be a way of salvaging the latter?

Two ways in which this might be possible suggest themselves. The first depends on allowing that non-designating terms might still occur in true identity statements. Someone who takes this view will, for example, think that it is a necessary truth that Hesperus is Phosphorus, even thought there are worlds in which the planet Venus does not exist.[10] Now, given that the most basic arithmetical statements, and no doubt the ones about which intuitions concerning necessity are strongest, have the form of equations – 7+5=12, say – these turn out to be necessary truths. What is not safeguarded on this approach is the necessity of *all* arithmetical truths. In particular, existentially quantified truths are contingent.

An alternative approach considers the intuition of necessity to track not arithmetical truth simpliciter, but arithmetical truth contingent on the truth of the axioms. So, for a true arithmetical statement ψ, we are supported in thinking that $\ulcorner PA\wedge \rightarrow \psi \urcorner$ is necessary. For most purposes, we don't

[9]Not all counter-mathematicals are in the same situation here. Assume there is at least one strongly inaccessible cardinal – is the thought 'there are no strongly inaccessible cardinals' in any interesting sense unthinkable? Does entertaining the supposition even require much effort?

[10]In Kripkean terms, this view collapses weakly necessary into strongly necessary truths.

need to distinguish between the necessary truth of these two statements, since they stand or fall together at every world at which $\wedge PA$ is true, that is every world except the nihilist world. At the nihilist world, however, the conditional is vacuously satisfied, whereas in general we cannot in general assume that ψ will be true. It is, on the view being suggested, conditionals such as $\ulcorner \wedge PA \to \psi \urcorner$ to absolute necessity attaches. Modal thinking about mathematics is implicity if-thenist in nature.

Either of these options affords the arithmetical contingentist, holding as she does that numbers exist only contingently, with a way of doing duty to intuitions about the necessity of mathematical truth.

3

Even if *some* mathematical objects, sets perhaps, exist contingently it might seem uncontroversial that at least *numbers* exist of necessity. The purpose of this note has been to unsettle this conviction. There is a perfectly coherent metaphysical outlook on which numbers exist only contingently, and this can be reconciled with the considerations usually thought to motivate arithmetical necessitism. We ought not to assume, therefore, that the necessary existence of numbers is a position that does not require philosophical support.

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