

## WHEN DO SOME THINGS FORM A SET?

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ABSTRACT. This paper raises the question under what circumstances a plurality forms a set, parallel to the Special Composition Question for mereology. The range of answers that have been proposed in the literature are surveyed and criticised. I argue that there is good reason to reject both the view that pluralities never form sets and the view that pluralities always form sets. Instead, we need to affirm restricted set formation. Casting doubt on the availability of any informative principle which will settle which pluralities form sets, the paper concludes by affirming a naturalistic approach to the philosophy of set theory.

1

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Plural logic is now a well understood tool of the philosophical logician. Taking its immediate motivation from Boolos, and having diverse applications in the philosophy of language, metaphysics, and the foundations of mathematics, the language of a plural logic admits variables which range over some things in plurality.<sup>1</sup> These are written ‘ $xx$ ’, ‘ $yy$ ’, and so on, and can be bound by quantifiers. So, for example, a formula of a typical such language might read,

$$(1) \quad \exists y \exists x x y < x x$$

This asserts that one thing is among some things, the (logical) predicate ‘ $<$ ’ being read ‘is among’. For the sake of brevity we talk about variables such as ‘ $xx$ ’ as ranging over *pluralities*. This is a convenient singularising locution, but might be misunderstood. Unlike a class or a set, a plurality is not a collective object, a single entity standing in the converse of a membership relation to some other entities; rather a plurality is some objects<sup>2</sup>. Every legitimate instance of class talk corresponds to some plurality. It seems, however, that not every legitimate instance of class talk corresponds to a set. We may talk of the class of ordinals ( $O_n$ ), but there is no set of ordinals. Set formation, then, at least on standard accounts of sets, is relatively demanding. The mere fact that a given plurality exists in no way entails that there is a set having as its elements all and only the members of that plurality.

**Definition 1.** A plurality  $tt$  forms a set iff  $\exists x \forall y y \in x \leftrightarrow y < tt$

<sup>1</sup>For an introduction, see [Linnebo, 2012]

<sup>2</sup>For most of the literature on plurals, a single entity counts as a special case of a plurality.

Hereafter we abbreviate, ‘ $t$  forms the set  $s$ ’ as ‘ $t \equiv s$ ’.<sup>3</sup> Following Linnebo [Linnebo, 2010], we can now ask: under what conditions is there a set such that some plurality forms that set? Note the similar form of this question to Peter van Inwagen’s *Special Composition Question* (SCQ): under what conditions is there a whole such that some things (a plurality) compose it? A number of answers are proposed in the literature to SCQ. The mereological *nihilist* answers ‘never’: all that exists are simples. The *universalist* about composition, exemplified in the person of David Lewis answers ‘always’: any objects whatsoever compose together a fusion; not only are there molecules, tables, and organisms, there are also trout-turkeys [Lewis, 1991, 81]. Finally, some philosophers answer ‘sometimes’. Van Inwagen himself thinks that some things form a whole iff they comprise a life [van Inwagen, 1990], whereas Effingham adopts what he terms an *eleatic* view, admitting only those fusions which are causally efficacious [Effingham, 2007]. There is not an accepted word for the position which professes restricted composition, but let us call this doctrine mereological *occasionalism*<sup>4</sup>.

It is noteworthy that all these answers to the SCQ find parallels in response to the question when a plurality forms a set<sup>5</sup>. That question itself is posed by Linnebo, and is the subject of a good deal of ongoing discussion [Linnebo, 2010].

The present paper explores in turn the nihilist, universalist, and occasionalist answers to the question when some things form a set. To signpost the conclusion: whilst occasionalism is, in my view, the correct answer to that question, it is a fairly deflationary answer: hopes for a metaphysically substantive account of which pluralities form sets are dim. Instead philosophers ought to consider epistemological questions about the justification for belief in the existence of proposed sets. To this sort of question, my response is naturalistic. Kripke once said that there is no mathematical substitute for philosophy. The present contention is a converse of sorts; there is no philosophical substitute for mathematics.

<sup>3</sup>This notation seems to be becoming standard in discussions of this area, is owing to Burgess [Burgess, 2008].

<sup>4</sup>I take it that the subject matters are sufficiently distinct for there to be no danger of confusion with occasionalism in the philosophy of causation, associated with Malebranche. Cameron uses ‘restrictivism’ as an alternative [Cameron, 2010, 9].

<sup>5</sup>In discussing mereological universalism, van Inwagen himself hints at the parallel: ‘[Mereological] universalism corresponds to a position about sets which almost everybody holds: In every possible world in which, for instance, Tom, Dick, and Harry exist, there also exists a set that contains just them. Nihilism corresponds to nominalism (about sets)’ [van Inwagen, 1990, 74]. I think van Inwagen is wrong in stating that ‘almost everybody’ assents to the set theoretic version of universalism. As we will see, this issues in a either a deeply unorthodox naive set theory or a restricted plural logic.

1.1. **Answer One : Never (Set theoretic nihilism).** One of the earliest articles on plurals in contemporary philosophical logic, Black's *The Elusiveness of Sets* [Black, 1971], proposes attention to plurals as a possible alternative to belief in abstract<sup>6</sup> sets. He is concerned to confront a persisting issue; as it finds expression on the first page of a current undergraduate textbook:

...it is very difficult to say what a set is. In the beginning a set was simply a collection, a class, or aggregate of objects, put together according to any rule we could imagine, or no rule at all. Of course, this doesn't give a *definition* of a set, since if we try to explain what a collection, class or aggregate is, we find ourselves going round in circles. [Cameron, 1998, 1]

What is a topic for comment on the part of the mathematician is an issue of concern for the philosopher of set theory. Noting the ubiquity of set-talk in educational institutions (he was writing at the peak of the 'new mathematics'), Black questions whether we really have a grasp of the concept *set*. At the very least we are owed some elucidation:

Paul Cohen said, 'By analyzing mathematical arguments, logicians have become convinced that the notion of "set" is the most fundamental notion of mathematics' [Cohen, 1966, 50]. One might therefore expect mathematicians and logicians to possess a firm concept of 'set'. But then they owe laymen and beginners – and philosophers, too – full explanation of a concept so fundamental and so important. [Black, 1971, 615]

Black is unimpressed with existing attempts. These view sets either as 'mysterious aggregations of distinct things into "unified" wholes' or as 'unknown things shared by certain coextensive properties'. The first approach is epitomised by Cantor's (in)famous description of a set as 'any assembly into a whole of definite and well-distinguished objects of our perception or thought', and is unacceptable as it stands, even once stripped of the constructivist undertones of 'our perception or thought'. This is because the nature of the 'assembly' in question is insufficiently clear for the description to be elucidatory. The second approach is also thought inadequate, since Black believes that no non-paradox-entailing and non-circular account is available which does adequate justice to the ability of human agents to grasp and work with the concept *set*.

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<sup>6</sup>In the metaphysician's sense of the word 'abstract'. A mathematician might be heard saying that the natural numbers form a set which is somehow less *abstract* than a certain  $\Sigma_2^1$  set of reals. She does not intend by this that I am likely to bump into countable sets whilst I am out doing my shopping.

1.1.1. *Black*. Black’s alternative is to regard set-talk as a conveniently disguised form of plural talk. Taking his lead from ordinary usage, he writes,

One primitive use of the word ‘set’ is as a stand-in for plural referring expressions. . . .  
If I say “A certain set of men are running for office” and am asked to be more specific, then I might say, “To wit, Tom, Dick, and Harry” – or, in the absence of knowledge of their names, I might abide by my original assertion. One might therefore regard the word ‘set’, in its most basic use, as an indefinite surrogate for lists and plural descriptions. [Black, 1971, 631]

It strikes me that ‘A certain set of men are running for office’ is an unnatural locution of English. Nonetheless, the example can be replaced easily enough (with, say, talk of sets of golf clubs, or the French ‘*ensemble*’), and a good feel for Black’s proposal thereby obtained. Black includes *prima facie* singular terms for sets (‘the Cantor Set’, ‘ $\{a, b\}$ ’) with, what he terms, ostensibly singular referring expressions. He goes on to account for the extension of set talk to talk of sets of sets (and sets of sets of sets, and . . .) by the invocation of (what are now called) superplurals and other higher-order plural terms.<sup>7</sup> Thus when I speak of  $\{\{a, b, c\}, \{a, b\}\}$  I am talking superplurally about  $a, b$ , and  $c$ , and about  $a$  and  $b$  in plurality. And when I talk about

$$\bigcup \{\{\{a, b\}, \{\{a, b\}\}\}, \{\{a, b\}, \{\{a, b\}\}\}, \{\{\{a, b\}, \{\}\}\} \dots\}$$

I am presumably using an  $\omega$ -th level plural on Black’s reckoning. It might be questioned whether ordinary users of mathematical concepts have access to any such infinite-order plural resources, a topic to which we will return<sup>8</sup>.

Before that, an important feature of Black’s approach should be emphasised. Black believes that plural talk cannot be made sense of in the cases of purported empty and single-membered pluralities. In spite of this, Black argues from the combination of this position on the limits of plurals and his philosophy of set theory to a scepticism about singletons and the empty set:

Of course, any transition from colloquial set talk to the idealised and sophisticated notion of making sense of a “null set” and of a “unit set” (regarded as distinct from its sole member) will cause trouble. From the standpoint of ordinary usage, such sets can hardly be regarded as anything else than convenient fictions (like the zero

<sup>7</sup>On these see [Rayo, 2006].

<sup>8</sup>It will turn out, of course, that if we want to understand set theory *as practiced* in Black’s terms, we require substantially more than  $\omega$ -th level plurals.

exponent in  $A^0$ ) useful for rounding off and simplifying a mathematical set theory.

But they represent a significant extension of ordinary use.

Black is by no means alone in thinking talk of the empty set and of singletons to be deserving of suspicion. Discussing the metaphysics of sets and the relation of these to pluralities, Oliver and Smiley cast doubt on the coherence of the notions of empty and single-membered sets [Oliver and Smiley, 2006]. On this basis they refuse to allow set theory a foundational rôle in relation to the rest of mathematics,

A set-theoretical foundation can only be provided by admitting the empty set and singletons, and, to use Quine's phrase, talk of them is troublesome. [Oliver and Smiley, 2006, 151]

We will turn to criticism of Black's position presently. Before that it is worth saying something in support of the characterisation of Black as a set theoretic nihilist. After all, Black doesn't require the mathematician to abandon her set theoretic discourse: doesn't this confirm that he Black is *not* a nihilist? No: Black will allow the set theorist her characteristic language precisely because he thinks that it does not incur ontological commitment to sets. There are no such things as sets: hence no plurality ever forms a set. Rather set theory admits paraphrase in plural terms. Properly understood, its subject matter is not abstracta picked out by the kind term 'set', but instead more common-or-garden entities considered in plurality. Hence the nihilism.

What are we to make of this? Black is proposing an elimination of set theoretic ontology in favour of an *ideological* commitment to plurals. In this respect his project is recognisably similar to that of those, following Boolos, who advocate the elimination of proper classes using plurals [Boolos, 1998]. There is a radical disparity, however, in the extent of the ideology required in the two cases. Uzquiano, a typical eliminator of classes, requires no more than simple plural quantification and reference, whereas the Black project requires an iterated hierarchy of plurals with as many stages as there are in the set theoretic hierarchy [Uzquiano, 2003]. Given the size of the smallest 'natural' model of ZF (and the smallest model *simpliciter* of ZF2), it is safe to state that Black will require at least  $\theta$ -th level plurals, for  $\theta$  strongly inaccessible. And this is potentially only the beginning. Depending on which large cardinal assumptions we wish to admit within our set theory, we

might need to avail ourselves of plurals of considerably higher level.<sup>9</sup>

1.1.2. *Plurals without end?* But are there any such resources of which to avail ourselves? Can we be confident that there are Woodin level plurals<sup>10</sup>? Or, more precisely, can we be *as confident* that there are Woodin-level plurals as we might be in some hypothetical future that there are Woodin cardinals? Can we even understand what there being Woodin-level plurals would involve without some grasp on an antecedently understood set theory? At this point in the dialectic, the issue becomes not simply whether the requisite plural resources exist, but also whether they are independent of set theory. If the latter condition is not satisfied, then the Black project of eliminating set theoretic commitments in favour of plurals will not succeed.

Some recent work of Linnebo and Rayo's is relevant here [Linnebo and Rayo, 2012]. They take as their starting point a quotation from Gödel,

only one solution [to the paradoxes] has been found, although more than 30 years have elapsed since the discovery of the paradoxes. This solution consists in the theory of types. . .

It may seem as if another solution were ordered by the system of axioms for the theory of aggregates, as presented by Zermelo, Fraenkel and von Neumann; but it turns out that this system is nothing else but a natural generalization of the theory of types, or rather, it is what becomes of the theory of types if certain superfluous restrictions are removed. [Gödel, 1995, 45-6]

Note that a plural logic with iterated levels of plurals is a version of type theory. Wishing to support Gödel's position, Linnebo and Rayo attack the commonly-held position that there *is* an important difference between set theory and type theory, in that the former (but not the latter) incurs ontological commitment to a specific type of entity - sets. By contrast, claims the near-orthodoxy, the commitments of type theory are purely ideological, and ideological commitments are less costly than ontological commitments.

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<sup>9</sup>A potential move here on the part of the nihilist is to rest content with recovering sufficient set theory to implement extra-mathematically *applicable* mathematics – reckoned by Michael Potter to be no more than the theory of sets of rank  $\leq \omega + 20$ . I find this unsatisfactory, since I do not think that philosophical accounts of mathematics should revise or restrict the mathematical enterprise. I note it here, however, for the less naturalistically inclined – although this strategy is of course hostage to the possibility of currently unapplied mathematics finding applications.

<sup>10</sup>That is,  $\kappa$ th level plurals, for  $\kappa$  a Woodin cardinal. On these see [Steel, 2007].

It seems to me that ideological commitments quite clearly *are* less costly than ontological in many cases. One example famous from the literature for which this principle appears to hold is that of the Cheerios in the bowl, considered in contrast to the set of those Cheerios. For an even more homely example, consider kittens. It is not very demanding to come up with a new way of talking about kittens, and thereby introduce a new ideological resource. On the other hand the introduction of a new kitten, an addition to ontology, is a fairly demanding matter, customarily requiring some exertion on the part of adult cats.

Nonetheless, whilst the orthodoxy might hold sway in these quotidian cases, might things change when we contemplate admitting weightier ideological resources? Linnebo and Rayo consider the admission of infinite types<sup>11</sup>. As we have seen, these will be required for the successful fulfillment of Black's project of reducing set theory to plural logic. Given two other conditions<sup>12</sup> the equivalence of type theory to iterative set theory can be proven, and Gödel's assertion in the above quotation confirmed. What are the consequences of this formal result for the issue of the relative costliness of ontological and ideological commitment? Linnebo and Rayo state their own view briefly,

Our own view is that it would be a mistake to eschew one of the hierarchies over the other. We would like to suggest instead that the two hierarchies (ideological and ontological) constitute different perspectives on the same subject-matter.

This gets developed by Linnebo and Rayo in terms of a particular meta-ontological position, wherein ontological commitment is 'lightweight' – a type theoretic statement incurs the same ontological commitments as its set theoretic equivalent, but this commitment is 'undemanding' on the world. Discussion at this stage would take us too far off-topic, but turns out to be superfluous, since there is a good argument – admissible across a range of views on the nature of ontological commitment – to be had that infinite-ordered type theory incurs ontological commitments equivalent to those of set theory<sup>13</sup>. So whether or not *Linnebo and Rayo's* argument to the effect that infinite type theory carries weighty ontological commitments is successful, there is another argument to those same conclusions.

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<sup>11</sup>Their motivations for doing this come from semantic theorising.

<sup>12</sup>Cumulativity and the admission of type-unrestricted predication.

<sup>13</sup>My own view is one of opposition to Linnebo and Rayo's metaontology. I suspect that some sense could be made of the suggestion that a commitment is lightweight in terms of a neo-Aristotelianism that views 'lightweight' commitments as grounded in more fundamental entities [Schaffer, 2009]. So, for instance, a lightweight commitment to the *Fs* might issue from a belief in *Fs* on the basis of our linguistic practice of referring to the *Fs*, the *Fs* themselves being ontologically grounded in these practices. I think that mathematical entities are poor candidates for being thus lightweight.

This argument goes as follows. How am I to write down (or, by analogy, otherwise express in language or thought) infinite types? In the finite case, representing the type is straightforward: I can either index type variables using natural numbers or, in explicitly plural systems, repeat letters to represent the level of plural: ‘ $xx$ ’, ‘ $xxx$ ’, ‘ $xxxx$ ’, and so on. Things are far less straightforward in the infinite case. The recourses that have served us well thus far will quickly become unavailable. Whilst we can avail ourselves of the natural numbers as indicies for a little longer – for example, by using the odd numbers to index finite level variables, and the even numbers to index infinite level variables, we will quickly ‘run out’ of natural numbers. Meanwhile, no repetition of letters in variables will ever get me as far as, let alone beyond,  $\omega$ -th level plurals. Sufficient for the task is a system of indexing variables *with ordinals*. Immediately we incur the ideological commitments that come with ordinal talk. I will now argue that we also incur an ontological commitment equivalent to that of the theory of the ordinals.

What does the type theorist need in order to make good the claim that she has a system equivalent to set-theory? To start off with, she has to be confident that for any stage in the cumulative hierarchy  $V_\alpha$  there are variables of type  $\alpha$ . She can certainly write down numerals for some very large ordinals, according to the standard nomenclature:

$$0 \dots \omega, \omega + 1 \dots \omega \cdot 2, \omega \cdot 2 + 1 \dots \omega^2 \dots \omega^\omega \dots \omega^{\omega^\omega} \dots \epsilon_0 \dots$$

But she will never write down (indeed, is physically constrained from writing down) indicies for all the intervening levels, indicated by the ellipses above. Not only this, but she will no doubt eventually run out of expressive resources as the types climb ever upwards. So what becomes of her claim that for every stage in the set-theoretic hierarchy there is a corresponding type?

Here is a way of defending that claim. The availability of variables of a given type is not hostage to tokens of such variables ever being written down, or talked or thought about. Rather, the variables exist quite apart from their concrete instances, and in virtue of this the equivalence of type theory to set theory is secured. A natural way of making the point is to describe the class, or plurality, of type variables as the minimal closure of the class/ plurality of type 1 variables under some generation rules (for any type  $n$  variable, the corresponding type  $n + 1$  variable exists; if variables exist of every type  $n$  strictly less than some limit  $\alpha$ , then type  $\alpha$  variables exist). This recourse clearly has *ontological* commitments to as many types of variables (or indicies for variables) as there are

ordinals<sup>14</sup>. The nihilist faces a dilemma: either she abandons the claim that there is a type for every stage in the set-theoretic hierarchy, in which case equivalence fails, or else she incurs massive ontological commitments which undermine her own motivation.

1.1.3. *Assessment of Black's project.* Black was attempting to show that our talk of sets can, and should, be analysed in plural terms, and therefore to defend set theoretic nihilism whilst allowing mathematicians their customary language. We have encountered good reasons to believe that he is unsuccessful. He lacks higher-order plural resources of the sort that would be required to obtain the cumulativity which is central to the mathematical notion of set. Even if he were equipped with such resources, it looks like Black will require ontological resources that undermine his project. Moreover, his rejection of the empty set and singletons suggest that, far from analysing what mathematicians in fact say and do, he is changing the subject, offering plurals as an *alternative* to standard sets, in spite of his own stated objective. Whatever mathematicians denote with the *prima facie* kind term 'set', it doesn't give the impression of being anything like the pluralities proposed by Black. We can say more: the whole strategy of seeking to understand set theory with reference to a pre-mathematical notion of set (one which Black, rightly, takes to be a plural notion), looks misguided. Set talk, as practiced in mathematics departments, is a highly specialised discourse drawing on substantial resources and far removed from everyday life. The two occurrences of 'set' in 'there is a set of golf clubs in the hall and a set of all countable ordinals in the hierarchy' are equivocal. This conceptual distinction will be of importance in the next section.

Let us turn to that section without further ado. Given that set-talk isn't simply a convenient means of talking about pluralities, the question remains: when does a plurality form a set? We consider an answer opposite to that of the nihilist.

1.2. **Answer Two : Always (Set theoretic universalism).** The set theoretic universalist holds that every plurality forms a set. That is to say, she accepts the principle Linnebo terms *collapse*:

(COLLAPSE)  $\forall xx \exists y xx \equiv y$

<sup>14</sup>Note that the point here is not that the nihilist is committed to ordinals as such – merely that she incurs commitments equivalent to those of the full theory of the ordinals inasmuch as they include a requirement that there be as many objects as there are ordinals, a requirement that is especially problematic for the nihilist (a key, but not the only, motivation for nihilism being the reduction of ontological commitment). This observation blocks a reply anticipated by a referee: isn't there an argument, parallel to that in the main body of the paper, to the effect that first-order logic is committed to the natural numbers, which it requires to index its variables? But, the reply continues, since this is absurd, so is the parallel argument to a conclusion about ordinals. I am not making the parallel claim. Rather a genuine parallel argument would conclude that someone invoking the language of first-order logic is committed to denumerably many objects, namely the variables of the language. This is both uncontentious (witness any competent logic textbook) and unproblematic.

In the light of Russell's paradox this strategy is hopeless within the bounds of classical logic if that logic includes a generous enough plural comprehension principle. For, once we admit the comprehension schema of **PFO+**, with the normal restrictions to prevent clash of variables,

$$\text{(COMP)} \quad \exists y \phi(y) \rightarrow \exists x x \forall x (x < x \leftrightarrow \phi(x))$$

a proof of Russell's paradox is easy. We'll consider in due course the strategy of restricting plural comprehension to preserve universalism. If that route is not taken, So adherence to pure universalism<sup>15</sup> involves the rejection of some classical principles. In fact, a *substantial* weakening of classical logic will be required; Russell's paradox can be proved in intuitionist logic. Two projects stand out as deserving of recognition, the dialethic approach of Priest and others, and Weir's formulation of naive set theory in a logic for which the transitivity of implication fails [Weir, 1998][Weir, 2005]. For reasons of brevity we'll confine our attention to dialetheism. Similar considerations to those assayed here in that case apply to Weir's view.

1.2.1. *Dialetheism and set theory.* *Dialetheism* is the view that there are *dialethias* – true propositions of the form  $\ulcorner (\phi \wedge \neg\phi) \urcorner$ . This opens the way for the embracing of naive set theory: the proposition that  $(r \in r \wedge r \notin r)$ , where  $r = \{x : x \notin x\}$ , for example can now be regarded as a dialethia.

For dialetheism to have any plausibility it must not entail *trivialism*, the doctrine that every proposition is true [Kroon, 2004]. In classical logic, we have:

$$\text{(Explosion)} \quad (P \wedge \neg P) \vDash Q$$

And the corresponding proof theoretic principle *ex falso quodlibet* – from a contradiction, infer anything you like. Thus classically, the supposition that the Russell set exists yields a proof that  $0 = 1$ . In order to avoid trivialism, dialetheism requires as a background logic a *paraconsistent logic*, for which (Explosion) fails. Typically such logics lack disjunctive syllogism, whether as a basic or derived rule<sup>16</sup>. The most familiar present-day systems of paraconsistent logic are Priest's propositional

<sup>15</sup>We will encounter presently an *impure* universalism, the modal universalism of Linnebo.

<sup>16</sup>With DS a proof is forthcoming of EFQ

*Proof.*  
 $(P \wedge \neg P)$   
 (2)  $P$  (1,  $\wedge$  E)  
 (3)  $P \vee Q$  (2,  $\vee$  I)  
 (4)  $\neg P$  (1,  $\wedge$  E)  
 (5)  $Q$  (3,4,DS)

and first-order versions of his **LP**, which may be understood semantically as the relevant logic **FDE** with the constraint on valuations that no formula is assigned the empty set [Priest, 1979]. More transparently, **LP** permits truth value *gluts* – some formula may be both true and false under some valuation – but not truth value *gaps* – every formula is at least true or false under every valuation. For further details see [Beall, 2004].

1.2.2. *Paraconsistent set theory.* Some work has been done on the formulation of naive set theory in paraconsistent logic. Priest and Routley list some of the features such theories allow for: ‘Boolean operations, ordered pairs, functions, power sets. . . the null set, infinite sets<sup>17</sup>’ [Priest and Routley, 1989, 373]. On this basis they assure us that ‘naïve set theory appears to provide for the set theory required by normal mathematics’. No gloss on ‘normal mathematics’ is supplied, but presumably the thought is that a naïve theory of sets, formulated in a paraconsistent logic, is adequate for the purposes of mathematicians working outside of set theory and logic themselves, and in particular for the needs of the mathematics that has applications in natural science.

The extent of *classical recapture* – how much of classical mathematics can be recovered within dialethic set theory – is a topic of ongoing research. In his [Brady, 1989] Brady shows that the theory is non-trivial. Both the continuum hypothesis and the generalised continuum hypothesis are open questions, as is much else besides.

1.2.3. *Paraconsistency and mathematical practice.* Of perhaps more pressing concern is the question whether a paraconsistent logic is adequate for the purposes of mathematical proof. The most immediately alarming feature of paraconsistent logic, when viewed from the perspective of textbook mathematics, is the failure of *reductio ad absurdum* to be valid. Consider here the canonical proof of the irrationality of  $\sqrt{2}$ .

**Proposition.**  $\sqrt{2}$  is irrational

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□

<sup>17</sup>With respect to the last, crucially important, provision for infinite sets, they offer a proof: Consider  $\{x : \exists y x \in y\}$ . This is mapped into a proper subset of itself by  $x \mapsto \{x\}$ , and so is infinite on the Dedekind definition.

*Proof.* Suppose towards a contradiction that  $\sqrt{2}$  is rational. Then for some  $a, b \in \mathbb{Z}$ , such that  $a, b$  are coprime,  $\sqrt{2}$  can be written irreducibly as  $\frac{a}{b}$ .

Manipulation gives us  $a^2 = 2b^2$ . Hence  $a$  is even. A little algebra gives us that  $b$  is also even. But this contradicts the claim that  $\frac{a}{b}$  is irreducible. Hence  $\sqrt{2}$  is not rational.  $\square$

There are other proofs of the proposition – geometric and analytic, for example – which likewise make use of *reductio*. Are we to rob mathematics of so basic a result, explicitly claiming instead – perhaps – that  $\sqrt{2}$  is rational (and also irrational)?

The dialetheist does not want to rob us of such fundamental mathematics, and needs to be able to account for the history of the development of mathematical thought; how it was that reasoning by *reductio* proved so fruitful and allowed humankind access to mathematical truth on such a reliable basis. Two things are required, then: some account of *which* contradictions are acceptably believed to be both true and false, and some explanation for *why* classical reasoning works so well so widely given that, for the dialetheist, it fails to capture the laws of logic. On the first point, Priest insists that dialethias are unusual; straightforward truth and falsity are the norm, and that dialethias occur with respect to parts of reality that are distinguished somehow, perhaps by generality (as in set theory), or by indeterminacy (as in cases of vagueness), or by some other feature (*ungrounded* sentences such as ‘this sentence is false’ providing a case in point<sup>18</sup>.) Rational acceptance of dialethias is, therefore, highly constrained: only in exceptional cases, where the cost of not accepting the dialethia would be substantial in terms of epistemic virtues other than consistency, should a contradiction be believed to be both true and false. This feeds into the response to the second point: precisely because good reasoning is *usually* classical, it is unremarkable that classical reasoning is *usually* good.

For all their ingenuity, these responses fail to convince. Suppose I am a Greek mathematician at the time of the discovery of irrational magnitudes. Not only my mathematics, but also my entire metaphysico-religious outlook, has as a central component the claim that all magnitudes are rational. Revising this belief is, for me, unthinkable. To do so would undermine my entire picture of reality. Whatever claims might be made, from a God’s eye perspective, about the eventual fruitfulness of such revision, the only fruit I could envisage it bearing are ones of confusion. Now suppose that I am

<sup>18</sup>Naive truth theory, captured by  $\forall P \text{True}(\ulcorner P \urcorner) \leftrightarrow P$ , with  $\ulcorner P \urcorner$  a name for  $P$ , perhaps obtained by arithmetisation techniques, can be formulated in **LP** such that the only dialethic instances of  $\text{True}(\ulcorner Q \urcorner)$  are for  $Q$  ungrounded in Kripke’s sense [Kripke, 1975]. For details see [Priest, 2002, s8]

presented with a geometric proof that there *are* irrational magnitudes. Am I not perfectly justified, in Priest's own terms, in taking a dialethic route out – there both are and aren't irrational magnitudes? If ever there was, in a reasoner's own terms, an exceptional case where the acceptance of a dialethia is justified, our Greek mathematician provides one. And yet our Greek mathematician is no mere fiction: I have instanced in her the pre-Pythagorean worldview. Why, we should ask Priest, should she not, in his own terms, have taken the dialethic route, and thereby caused swathes of modern mathematics to be still-born, dependent as it is on the wholehearted acceptance of the conclusion of this epochal proof by *reductio*? The response comes back that, as with any science, the validation of good mathematical reasoning is in large part *post eventum*, owing to the fruitfulness of accepting the conclusion of that reasoning for subsequent discovery. In this case, the classical response to the proof, it would be insisted, has been more than warranted. Indeed so, but our original question has not been answered. *How*, given that logic is paraconsistent, and that logic is a guide to reasoning, could this very mathematician have been right in accepting *reductio* in this case, as Priest agrees she was?

The loss of disjunctive syllogism also looks as though it will be a major cost. Burgess<sup>19</sup> offers us the example of an imagined mathematician, Wyberg, who has been working on von Eckes' conjecture, and has come across the result of one Professor Zeeman, that for every natural number  $n$ ,  $A(n) \vee B(n)$ . He proceeds to prove von Eckes' conjecture [Burgess, 1981, 101]:

He writes a set of notes, 'A Proof of von Eckes' Conjecture' with the following structure: First comes his proof that  $\neg A(1)$ . Second comes a linking passage:

'And so we see that the MacVee Conjecture fails. Now Zeeman has recently announced the result that for all  $n$ , either  $A(n)$  or  $B(n)$ . Hence we must have  $B(1)$ . We now proceed to put this fact to good use'.

Third follows the derivation of von Eckes' conjecture from  $B(1)$ .

Thus an absolutely typical application of disjunctive syllogism, characteristic of everyday mathematical practice. And again, for reasons discussed above with respect to *reductio*, it seems that the dialetheist will have her work cut out attempting to explain the fruitfulness of this rule, failing as it supposedly does, to capture anything logical.

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<sup>19</sup>Burgess' target is the claim that relevant logics provide a satisfactory formalisation of mathematical reasoning. His immediate focus is the systems of Anderson and Belnap, but the same considerations apply to Priest's LP.

1.2.4. *Answer Two-and-a-half: Potentially Always (Modal universalism)*. Linnebo considers (COLLAPSE) to be an attractive principle:

The view... has great intrinsic plausibility. If it were not for [the paradoxes] we would probably all have believed it. Why should some things  $xx$  not form a set? The semantics of plural quantification ensures that it is determinate which things are among  $xx$ . And a set is completely characterized by specifying its elements. We can thus give a complete and precise characterization of the set that  $xx$  would form if they did form a set. What more could be needed for such a set to exist? [Linnebo, 2010, 3]

This line of thought is not irresistible. There is some logical space between the epistemological view (enshrined in Extensionality) that ‘a set is completely characterized by specifying its elements’ and the metaphysical view that there is *no more* to a set than its elements. According to one popular family of positions on the nature of sets, dubbed by Lewis the *lasso hypothesis* [Lewis, 1991, 42], a set consists in the elements together with some collectivising entity (the so-called lasso). If one of these positions is correct, whilst I would know which set some  $xx$  would form, *if* they formed a set<sup>20</sup>, there remains the further question, whether they form a set, that is whether the salient collectivising entity exists<sup>21</sup>. It is, presumably, the job of set theory to offer answers here.

Leaving that issue aside, Linnebo is unwilling to abandon classical logic, and consequently accepts that (COLLAPSE) is false<sup>22</sup>. He reconciles this with the view that (COLLAPSE) is intuitively well-motivated, by claiming that set theoretic quantifiers are implicitly modalised, ‘ $\forall$ ’ and ‘ $\exists$ ’ abbreviating ‘ $\Box\forall$ ’ and ‘ $\Diamond\exists$ ’ respectively.

(COLLAPSE $^\diamond$ )  $\Box\forall xx\Diamond\exists y xx \equiv y$

The modality operative here is not intended to be metaphysical modality – Linnebo believes that if (pure, at least) sets exist, they do so of metaphysical necessity. Whilst arguing that all that is required of the modality at issue is that it be ‘suited to explicating the iterative conception of

<sup>20</sup>It is a nice question how one would go about assessing the truth of the counterfactual here.

<sup>21</sup>Compare here, Cameron’s comments on mereological universalism in the context of claims about composition-as-identity [Cameron, 2010, 19].

<sup>22</sup>And, it should be added in the light of the preceding section, not also true!

set' [Linnebo, 2010, 15], Linnebo proposes an account in terms of the individuation of mathematical objects:

To individuate a mathematical object is to provide it with clear and determinate identity conditions. This is done in a stepwise manner, where at any stage we can make use of objects already individuated and use this to individuate the set with precisely those objects as elements. A situation is deemed to be possible relative to one of these stages just in case the situation can be obtained by some legitimate continuation of the process of individuation. [Linnebo, 2010, 16]

We will return in a moment to the question how satisfactory this is. First we examine how Linnebo uses (COLLAPSE<sup>◇</sup>) in combination with plural logic and some basic ideas analytic of the concept set to provide a foundation for set theory.

1.2.5. *Modal set theory.* In his *The Potential Hierarchy of Sets*, Linnebo develops a modal set theory. The background modal logic is **S4.2** - the accessibility relation of the model theory for which Linnebo believes to capture the principles governing the individuation of sets<sup>23</sup>. To plural<sup>24</sup> **S4.2** with identity<sup>25</sup> Linnebo adds the principle (NecInc)<sup>26</sup>:

$$(2) \quad x < xx \rightarrow \Box x < xx$$

And its natural partner:

$$(\text{NecNonInc}) \quad x \not< xx \rightarrow \Box x \not< xx$$

Along with a principle which rules out pluralities acquiring new members if the domain expands:

$$(\text{CL-}<) \quad \forall x (x < xx \rightarrow \Box \theta) \rightarrow \Box \forall x (x < xx \rightarrow \theta)$$

<sup>23</sup>**S4.2** results from adding to **S4** the axiom G:  $\Diamond \Box P \rightarrow \Box \Diamond P$ .

<sup>24</sup>Linnebo admits an empty plurality, and modifies the plural comprehension schema accordingly.

<sup>25</sup>Including the axiom  $x \neq y \rightarrow \Box x \neq y$ .

<sup>26</sup>Because Linnebo is clear that he does not take his modal operators to indicate metaphysical modality, the considerations assayed against (NecInc) in [Hewitt, 2012b] do not apply.

Linnebo calls the resulting system **MPFO**. Any wff of the plural language **PFO** has a potentialist translation in **MPFO**, obtained by replacing all of its quantifiers with the relevant modalised quantifiers ( $\Box\forall$  and  $\Diamond\exists$ ). We write “ $\phi^\diamond$ ” for the potentialist translation of  $\phi$ . Hence, for instance,  $(\text{COLLAPSE}^\diamond)$  is the potentialist translation of the inconsistent principle  $(\text{COLLAPSE})$ .

Linnebo goes on to formulate a theory of the nature of sets, NS, in **MPFO**. The axioms of NS are:

$$(\text{Ext}) \quad x = y \leftrightarrow \forall u (u \in x \leftrightarrow u \in y)$$

$$(\text{ED-}\in) \quad \exists yy \Box \forall u (u < yy \leftrightarrow u \in x)$$

$$(\text{F}) \quad \Box \forall x [\exists y (y \in x) \rightarrow \exists y (y \in x \wedge \forall z (z \in x \rightarrow z \notin y))]$$

$$(\text{ED-}\subseteq) \quad \exists xx \Box \forall u (u < xx \leftrightarrow u \subseteq a)$$

A word on the intended import of these axioms: (Ext) captures the thought, frequently taken to be analytic of the concept *set*, that sets are extensional. (ED- $\in$ ), in the context of the usual semantics for plural logic, expresses the extensional definiteness of set-theoretic membership; (ED- $\subseteq$ ) does the same for subsethood. (F) captures the well-foundedness of membership<sup>27</sup>.

From  $\text{NS} \cup \{\text{COLLAPSE}^\diamond\}$  the potentialist translations of all the axioms of ZF, save Replacement and Infinity can be proved in MPFO. Linnebo recovers these by invoking two further principles. First, he takes limitation of size to be implicit in our ideas about the nature of sets, justifying (where  $\ulcorner \text{FUNC}(\psi^\diamond(u, v)) \urcorner$  abbreviates the claim that  $\psi^\diamond(u, v)$  is functional):

<sup>27</sup>Is well-foundedness part of our basic concept of set? Linnebo doesn't need to provide a decisive answer here: if challenged, he can retort that (NS) captures our basic ideas about well-founded sets, without prejudice to the issue of whether there are other sets. Note that the presence of (F) in NS makes the subsequent recovery of the ZF axiom of Foundation trivial.

$$(ED\text{-REPL}) \quad FUNC(\psi^\diamond(u, v)) \rightarrow \Box \forall x x \diamond \exists y y (\forall u < x x) (\exists v < y y) \psi^\diamond(u, v)$$

In combination with NS and  $(COLLAPSE^\diamond)$  this allows a proof of the potentialist translation of Replacement. In order to recover Infinity, Linnebo applies a reflection principle:

$$(\diamond\text{-REFL}) \quad \phi^\diamond \rightarrow \diamond \phi$$

Linnebo calls the modal set theory which has MPFO as its background logic and consists of NS, together with  $(COLLAPSE^\diamond)$ , every instance of  $(ED\text{-REFL})$  and every instance of  $(\diamond\text{-REFL})$  MS. It is consistent if ZF is.

We can readily dispel any thought that MS can supply us with the tools for settling many (if any) questions about the height of the hierarchy unsettled by ZF (or ZF2). It is an easy corollary of Linnebo's consistency results and the Second Incompleteness Theorem that MS cannot prove the existence of even a worldly cardinal.

1.2.6. *What is the modality at work?* The technical interest of Linnebo's modal development of set theory may be granted readily. For our purposes, though, the question is whether the demonstration that (a potentialist version of) our best accepted set theory can be derived within MS gives us reason to be confident that modal universalism is a viable answer to the question of set formation. This, in turn, depends on the philosophical interpretation given to the formalism. Of central importance here is whether an understanding of the modality formalised by ' $\Box$ ' and its dual is available which serves the foundational purposes at hand.

As we've already seen, Linnebo has a favoured reading of the modal operators for the purposes of modal set theory. On this reading it is possible that  $\phi$  just in case I can individuate further mathematical objects such that  $\phi$ . What is meant by 'individuation'? Just this: I can individuate an  $x$  such that  $\phi(x)$  iff there is a possible extension of my language such that there is a singular term ' $a$ ' such that ' $\phi(a)$ ' is true. Notice here that there is a very close connection between language, specifically the singular terms occurring true declarative sentences, and ontological commitment. This has a distinctly Fregean feel. Merely marking such a connection does not decide between a 'lightweight' or a 'demanding' approach to meta-ontology. The proponent of the latter account will insist that,

perhaps because it takes a good deal of co-operation from the world for sentences to be true, or perhaps because extending our language significantly is difficult, ontology is not cheap. The supporter of the former approach will insist that the close relationship of ontology to language demonstrates just how easily expanded are our ontologies.

It is not clear to me that Linnebo is entitled to assume that we can go on individuating ontology so indefinitely that we can confidently assert as true the modal translations of the axioms of set theory, on his reading on the modal operators. The same issue about expressive resources that arose regarding large plural hierarchies in our discussion of nihilism recurs here. Concerns about the motivating metaontology also impress themselves upon us. Are we really prepared to believe that, leaving aside atypical cases such as linguistic entities, which entities there are can be adjudicated solely in virtue of attention to language? This is an implausibly anthropocentric approach to ontology, and one which is in grave danger of collapsing into anti-realism. Take some oxygen atom, call it  $a$ . Now, I submit,  $a$  exists quite apart from our ability to individuate it, or from the existence of beings capable of individuating it. Whether or not there are oxygen atoms has nothing to do with there being agents possessing the sortal concept ‘oxygen atom’ and the capacity to individuate entities falling under it. As, for oxygen atoms, so for sets, at least in the absence of solid motives for believing otherwise<sup>28</sup>.

There is a competing account of the sort of modality at work in Linnebo’s system **MS**. Modal logic is often understood, particularly in pure mathematics and computer science, as the mathematical study of relational structures<sup>29</sup> Take some things, and some relations on those things: modal logic permits us to study the resultant systems model theoretically – let the nodes (worlds) be the things and the relation(s) be the accessibility relation(s) on nodes. One such relational structure is the cumulative hierarchy  $V$  under the relation that  $V_\alpha$  bears to all and only the  $V_\beta$ , for all  $\beta > \alpha$ . Now, of course, we can use modal logic to study this structure; and this, so the sceptical response to Linnebo goes, is precisely what Linnebo’s project should be understood as doing. The project is not devoid of interest: it is potentially fruitful, and certainly bolsters the claim that ZF captures the iterative conception of set. What it does not do, however, is offer the kind of foundational metaphysical insight into the set theoretic universe that Linnebo promises. What stands in need of explanation from the point of view of this sort of metaphysical undertaking, naming the cumulative hierarchy, is

<sup>28</sup>For a discussion of an anthropocentricity criticism against a position similar to Linnebo’s see [Hale, 2013]

<sup>29</sup>See, for instance, [Blackburn et al., 2002].

assumed, rather than explained.

1.2.7. *Against the naïve intuition.* We have called the supposition that the principle expressed by (COLLAPSE) is somehow implicit in our concept of sethood, such that we have *prima facie* justification for believing it *the naïve intuition*. What do its supporters have to say for it? We've already encountered Linnebo's suggestion that because a set is entirely specifiable in terms of its elements, the existence of the elements suffices to secure the existence of the set. Once one understands what a set is, on this view, universalism follows. Against this, we maintained that a confusion is being traded on: given the existence of the elements, there is no ambiguity as to *which* set they would form, *if* they were to form the set. It is another thing altogether *whether* they form a set. However, Linnebo's is not the only argument in support of (COLLAPSE) having a basic hold on us.

Priest thinks that the naïve intuition is not only natural, but implicit in the possibility of even orthodox set theory. Priest accepts, what Hallett terms, *Cantor's Domain Principle*. Hallett describes this thus,

In order for there to be a variable quantity in some mathematical study, the 'domain' of its variability must strictly speaking be known beforehand through definition. . . this 'domain' is a definite, actually infinite series of values. [Hallett, 1984, 25]

An advocate of plural logic can accept Cantor's Domain Principle readily and unproblematically. A domain can be a definite, actually infinite, series of values, without the domain itself being an entity (and so, an available value). It is the reification of the set theoretic domain itself which leads us down the road to paradox. However, this fatal course can be circumvented by understanding the domain as some objects (rather than an object). This is the achievement, for instance, of Rayo and Uzquiano in [Rayo and Uzquiano, 1999]. Priest, however, ignores the plural option and insists that the practice of set theoretic quantification requires a universal set,

According to Cantor's Domain Principle. . . any variable presupposes the existence of a domain of variation. Thus, since in ZF there are variables ranging over all sets, the theory presupposes the collection of all sets,  $V$ , even if this set cannot be shown to exist in the theory. [Priest, 1995, 158]

This enlisting of naïve sets for a task which could be performed unproblematically by plurals is suggestive for a response the plural theorist can make to the advocate of the naïve intuition. This

is that the supposed intuition rests on a confusion between sets and pluralities<sup>30</sup>. These correspond to distinct forms of collection. Sets are distinct objects over and above their elements, and naive comprehension does not hold for sets. Pluralities on the other hand, are not distinct objects over and above their members – a plurality just *is* its members – and naive comprehension does hold for pluralities. If one fails adequately to distinguish between pluralities and sets, this can lead to the acceptance of naive comprehension for sets, and the belief that this acceptance is somehow intuitive, on the basis of intuitions about pluralities. Confusion is rendered especially likely by the natural language use of ‘set’ to denote pluralities – as in ‘the set of cutlery’ – and by the invocation of broad notions, such as that of *collection*, which are ambiguous between sets and pluralities. This confusion seems to have been present from the birth of set theory. As Kreisel says,

...naively sets present themselves in a number of distinct contexts (finite collections of concrete objects before us; sets of natural numbers satisfying more or less explicit conditions; sets of points in geometry.) One may therefore doubt whether any definite general notion (of set) is involved here; it looked more like a mixture of notions. As a matter of historical fact this was the common feeling among Cantor’s contemporaries. [Kreisel, 1969, 93]

On our present hypothesis, there is no general notion of set involved in all these cases. Rather, some collections are sets, some are pluralities. Pluralities conform to naive comprehension. Sets do not, as Russell showed. A distinction Stenius makes between *sets-of-things* and *sets-as-things* is helpful here. There are two senses of the English word ‘set’. One – sets *of* things – corresponds to our pluralities. The other – sets *as* things – is the present-day mathematico-philosophical usage, and corresponds to our sets:

It becomes disastrous for intellectual clarity if we believe that sets-of things are ‘things’. Whenever a set-of things is made into a whole, we introduce something new, which is *not* the mere set-of things. [Stenius, 1974, 168]

Not only for intellectual clarity, we might add, but also for theoretical consistency.

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<sup>30</sup>A comment on the dialectic here is in order. The diagnosis of confusion is not supposed to convince a committed universalist, who will typically minimise (or even eliminate) the distinction between sets and pluralities. Rather, it provides a means for the person antecedently convinced of the falsity of universalism (because, for example, of the costliness of rejecting classical logic) with a way of strengthening her case by offering an explanation of the naïve intuition. It may well also exert some dialectical traction on the undecided.

1.2.8. *Entanglement.* An objection raises its head. My contention thus far has been that the universalist confuses pluralities and sets. Might instead the confusion, in fact, be on my part? For I have been assuming that we can have an understanding of plurals antecedent to a grasp of set theoretic principles. In particular, I take it that we are in a position to accept every instance of (COMP) without admitting its inconsistent set theoretic cousin<sup>31</sup>,

(BAD)  $\exists x \forall y (x \in y \leftrightarrow \phi)$

But what if legitimate acceptance of (COMP) depends on acceptance of (BAD)? What if, as a referee suggests, we believe this to be so because we take ‘the concept of a plurality to be no better understood than a set’? Then, assuming classical logic, I do not escape Russell’s paradox and am in no better position than I suppose the universalist to be. Perhaps, then, I can make peace with the universalist by agreeing to a restriction of plural comprehension, agreeing that every plurality forms a set, whilst denying that there is any such plurality as the plurality of all and only the non self-membered sets<sup>32</sup>.

My response is that a restriction of plural comprehension results in a logic that does not permit quantification over all the pluralities. Some pluralities are quite deliberately excluded, including ones which appear to be denoted in textbooks - the ordinals, the non self-membered sets and so on<sup>33</sup>. Yet since my concern is with a metaphysical, and therefore an absolutely general, question I want to assert the existence of as many pluralities as possible, given only that there are such pluralities.

This needn’t, in fact, involve commitment to the position that plural quantification is absolutely determinate. Following Florio and Linnebo, we could supply a Henkin-style semantics for **PFO+**, thereby allowing the plural ‘domain’ to vary for a fixed first-order domain<sup>34</sup>[Florio and Linnebo, 2015].

<sup>31</sup>Which we may take as having the usual restrictions on variables in  $\phi$ .

<sup>32</sup>On restrictions on plural comprehension as a response to paradox see [Yablo, 2004].

<sup>33</sup>Of course, it is typical to understand these denotations in terms of proper classes, perhaps then reduced to meta-theoretic schemata. But in the spirit of [Oliver and Smiley, 13] and Boolos, I want to insist that this is a mistake. Mathematical English uses plurals routinely – ‘global choice implies that the sets are bijectable onto the ordinals’ – and there is no reason, apart from singularist prejudice (which is no *good* reason) not to take this use at face value.

<sup>34</sup>The resulting logic is complete, compact, and possesses both upward and downward Löwenheim-Skolem properties. As a consequence we lose a nice feature of plural systems with respect to set theory, namely that (given the, obvious, existence of a pairing function on the intended domain) we can get a  $\kappa$ -categorical axiomatisation of a system equivalent to ZF2. Kreisel’s appeal to second-order set theory in arguing for the determinacy of CH can therefore get support from the plural logician [Kreisel, 1967] [Weston, 1976]. A result of McGee’s extends this to the determinacy of all questions about the height of the hierarchy, given an *urelemente* set and the availability of absolutely general quantification [McGee, 1997]. This being so, I would like to defend the legitimacy of standard plural semantics, but this task is beyond present scope .

Such a semantics does require, however, that it is sound for every instance of (COMP)<sup>35</sup> And it is *at this point* that our sceptical interlocutor will protest: why believe, minus a tacit appeal to naive set theory, that every instance of (COMP) is *true*?

Two points can be made in reply. First, the complainant's position *is* a strongly sceptical one. In pushing us to reject instances of the plural comprehension schema they are asking us to entertain the position that in spite of our language being able to express some condition, we cannot talk about the things that satisfy that condition. This is surely surprising and the burden of proof arguably ought to rest with the person claiming that we cannot say of things collectively what we can say of each of them singularly. Second, an appeal can be made by the defender of the usual comprehension schema to a holist methodology, of the sort recently defended by Williamson [Williamson, 2013]. The schema is simple and elegant, it affords us logical resources which promise to do a lot of work in the foundations of mathematics, metaphysics, and other areas. Thus, by the usual criteria of theory choice (which, by holist hypothesis apply in logic and mathematics, just as in other sciences), there is pressing reason to accept the schema, even if the resultant loss of a set for every plurality is a genuine cost. So even *if* the loss of universalism is a cost, and I have suggested that there was never anything to be lost in the first place, that naive 'sets' so-called are pluralities misidentified because of conceptual confusion, there is an abundance of benefits on the other side of the scales.

These considerations will not, of course, convince the determined proponent of the view that our plural concepts are entangled with set theoretic content, and ongoing debate will quickly descend into intuition trading – a Boolosian assertion that I can *obviously* talk plurally about the Cheerios in a bowl, and for that matter about subpluralities thereof, without somehow implicitly commanding set theoretic knowledge, being deployed on one side, an incredulous stare at the impredicative plural comprehension scheme returned on the other. Joining this unproductive exchange will detain us unduly. There is always a limited response available to the persistent sceptic. What can at least be offered is an alternative *picture* to the sceptical one: the picture on offer is this, we have plural resources available to us as competent speakers of a natural language, before we enter the logic classroom. These admit of, and need no, genuinely informative explication in non-plural terms. Plural logic, with the usual comprehension schema, is simply a regimentation and extension of these pre-existing resources, that speakers who have not been drilled into translating plural locutions as

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<sup>35</sup>This is the *faithfulness* condition familiar from the semantics of second-order logic. For details see [Shapiro, 1991].

singular talk about sets will readily recognise as such. The system thus arrived at can be used to talk about sets, without prejudging the relation between these and pluralities.

1.2.9. *Category theory*. Priest suggests a motivation for naïve set theory that will require a little more work to fully address, even after the naïve intuition has been defused. He thinks that unorthodox sets are required for the purposes of category theory<sup>36</sup>. He is quick to dismiss the suggestion that proper classes are up to the task of supplying a foundation for category theory which can deal with large categories such as **Group** (which we define as the collection of all groups together with the collection of all and only the homomorphisms on groups closed under composition). Typically, collections here are understood as *sets*, so an arbitrary category  $C$  is defined as  $C = \langle Ob, Morph \rangle$ , for  $Ob$  a non-empty set, and  $Morph$  a set of morphisms on  $Ob$ . But now there is a problem: in the case of **Group**, for example, the required set doesn't exist – both  $Ob$  and the elements of  $Morph$  are too big to be sets. Invocation of proper classes alone will not suffice to avoid the difficulty, since the too-big-to-be sets collections are, on standard treatments of category theory, themselves elements of other collections. Once we go on to form, for instance, the category of functors between two categories, we get iterated membership to an extent that the invocation of proper classes to handle large categories<sup>37</sup> is in clear trouble:

though [proper classes] may allow us to conceptualise large categories, since they can not be members of other collections, it does not allow us to operate on them, form functor categories etc. The only way out is to admit that proper classes can be members of other collections. However, this does not solve the problem but merely undermines the weakness of the notion of proper classes. [Priest, 1987, 43]

The way out he minutes has indeed been advocated. Muller notes the availability of Ackerman's class theory, which allows classes to be elements, for the purposes of category theory [Muller, 2000]. As Priest argues, however, this seems to undermine the motivation for accepting proper classes, as a distinct type of set-theoretic entity, in the first place. The proponent of the kind of augmented plural logic developed in, for example, [Hewitt, 2012a] has options for supplying an account of categories which does not require classes. The combination of iterated plurals and ordering will allow us to simulate the set-theoretic structures customarily thought to be required to found category theory.

<sup>36</sup>On category theory see [Cheng, 2008]. Of course, it might well be argued that the thought that category theory requires a foundational account of its subject matter issues from a misunderstanding of category theory, in which case Priest can be answered swiftly. Thanks to Neil Barton for pressing this point in discussion.

<sup>37</sup>A category is defined as *small* if  $Ob$  and  $Morph$  are both sets; *large* otherwise.

To sum up: set theoretic universalism is unattractive. It entails paradox within a classical framework, and its supposedly intuitive status rests on a confusion between distinct collection concepts. We have already distanced ourselves from nihilism. The only remaining answer to the question under what conditions a plurality forms a set is the occasionalist one. To that our attention now turns.

**1.3. Answer Three : Sometimes (Set theoretic occasionalism).** Some things form a set; some other things don't. The empty set and its singleton, for example, form a set, whereas the ordinals don't form a set. Therefore both nihilism and universalism are false. Can we say more than this? There would certainly be something satisfying about being able to supply explanatory necessary and sufficient conditions for set formation. We'll look at two candidates before broadening out the discussion of occasionalism.

**1.3.1. The iterative conception.** The iterative conception of set is familiar amongst mathematicians and philosophers alike [Potter, 2004, Ch. 3][Boolos, 1971]. Begin with a stage  $V_0$  which may contain some non-sets, or may be empty. After every  $V_\alpha$ , there occurs a stage  $V_{\alpha+1}$  which consists of all and only sets that can be formed from entities occurring at stages  $V_\beta$  for  $\beta < \alpha$ , plus the non-sets (if any). At  $V_\gamma$  for  $\gamma$  a limit ordinal, *all* sets of *all* entities from earlier stages are formed, and the non-sets recur. Now consider the claim

**IT-SET:** Some  $xx$  form a set iff there is some  $\alpha$  such that the  $xx$  all occur at  $V_\alpha$ .

(IT-SET) is occasionalist. Let  $aa$  be the plurality consisting solely of  $a$  and  $b$ , both *urelemente*;  $\{a, b\}$  will be formed at  $V_1$ . On the other hand, at no stage will the set of all sets be formed. One might object to (IT-SET) on the grounds that one believed it not to be true. Suppose I believe that there is a universal set, but am not a universalist – I might accept Quine's set theory NF, for example. Then I will reject (IT-SET). There is a broad consensus that this position lacks motivation, and I report it here merely for the sake of comprehensiveness. More details can be found in [Forster, 1992]<sup>38</sup>.

A more popular reason for rejecting (IT-SET) than wholesale acceptance of the Quinean approach is that (IT-SET) is felt to be too restrictive, in that it rules out non well-founded sets<sup>39</sup> Theories which admit non well-founded sets, equiconsistent with ZF, are described in Aczel's [Aczel, 1988]. These modify ZF by removing Foundation, and adding some anti-foundation axiom securing the existence

<sup>38</sup>For recent dissent see [Hossack, 2013].

<sup>39</sup>In virtue of the well-ordering of the ordinals.

of non well-founded sets. They are not restrictive with respect to the uses of set theory in mathematical contexts. There are two options for philosophical defence of (IT-SET) in response to the challenge from believers in non well-founded sets. A quietist rejoinder simply maintains that a set is, by definition of the word ‘set’, an entity which occurs at some stage in the cumulative hierarchy: whether or not there are in fact set-like entities which are non well-founded is beside the point; (IT-SET) is an analytic truth. A more substantive reply appeals to the metaphysical dependence of sets on their members. A set, on this thought, depends on its elements for its existence, but the converse does not hold. I ground my singleton in being, my singleton does not ground me. This relationship of dependence, the thought goes, explains the progression of stages in the hierarchy, and thereby the correctness of the iterative conception of sets. Some philosophers, [Incurvati, 2011] for example, doubt that sense can be made of the notion of metaphysical dependence. I myself find it clear enough for working purposes: it is the converse of the relation Aristotle thinks that I bear to the form of having brown hair, or which theists believe God bears to everything else. One might very well dissent from these philosophical doctrines: to claim, however, that they don’t *make sense* renders it puzzling how philosophers continue to debate them, and strikes me as a prejudice inherited from positivism. We don’t need to pursue this debate though<sup>40</sup>, since even if the metaphysical dependence of sets upon their elements can be made sense of, a further worry arises: does this secure the iterative conception of set, and in particular well-foundedness? Why should we suppose that the grounding relation is itself well-founded? For the purposes of moving the debate on<sup>41</sup>, it is prudent to pursue the quietist response.

Far more philosophers than those who *reject* (IT-SET) accept the principle but doubt that it represents an explanatory answer to the question of set formation. Merely to say that all and only the sets which are formed at some stage exist seems to leave a lot open: in particular, the answer appears compatible with a wide variety of positions concerning the *width* and *height* of the variety. We have said that *all* possible sets of objects from earlier stages are formed at every stage, but how rich are the possibilities? And how far do the stages extend? Is there, for example, a stage  $V_\delta$ , for  $\delta$  supercompact?

<sup>40</sup>The discussions in [Incurvati, 2011] and [Potter, 2004] are a good basis for exploring the issues around dependence further investigation.

<sup>41</sup>This is not, of course, to deny that there are interesting and important metaphysical debates here. I myself am profoundly sceptical of the claim that there could be non well-founded grounding.

Formally, these questions correspond to the question which set theoretic axioms are licensed by the iterative conception. The *locus classicus* of discussion here is Boolos' [Boolos, 1971]. In this paper Boolos develops a formal stage theory, intended to capture the iterative conception of set, and proceeds to derive a number of the standard axioms of set theory. This procedure delivers all the axioms of Zermelo set theory minus Extensionality, that is all the axioms of (first-order) ZF minus Extensionality and Replacement. The non-derivability of Extensionality is, thinks Boolos, of no concern, since Extensionality is effectively definitional of sethood. Were it not for Quinean scruples about analyticity, he says, we would consider its truth analytic [Boolos, 1971, 27-8]. Replacement, however, is not forthcoming. That Replacement incarnates a thought about sets additional to the iterative conception receives support from the fact that Linnebo needs to import an additional reflection-like principle to NS (which looks very much like it captures an iterative notion) in order to permit the proof of its modal translation in MS.

1.3.2. *Limitation of size.* What principle concerning sets might motivate Replacement? The axiom ensures that any plurality which is no larger than some set itself forms a set. A common thought, then, is that the axiom captures a principle of *limitation of size* [Potter, 2004, 227-30]. Related to this is frequently a diagnosis of the set theoretic paradoxes: that these result from the attempt to form sets that are 'too big' – of the sets themselves, the non self-membered sets, the ordinals, and so forth. Certainly, the locution 'too big' has entered the mathematical vernacular with respect to such cases.

Might these considerations not motivate a necessary and sufficient condition for set-formation?

**(LOS)** A plurality forms a set iff it has fewer members than there are ordinals.

Is (LOS) true? The right-to-left direction does not appear problematic. It is difficult to see what could be troublesome about a plurality fewer in number than the ordinals forming a set. What about the left-to-right direction? Are all sets formed from pluralities fewer in number than the ordinals? If one is minded to think that the paradoxes result from 'big' collections then this will recommend itself as a working hypothesis. Yet consider the following case: there are as many *urelemente* as there are sets. Why should we rule out there being a set consisting of all and only the *urelemente*?

Consider the situation abstractly, apart from any developed set theory<sup>42</sup>. What trouble could be incurred by admitting this set? It lacks a key feature of both the Russell and Burali-Forti collections, self-membership (a feature which, incidentally, might give one reason to wonder whether the blame for paradox in these cases rests more at the door of non well-foundedness than of size). Naively, it is not apparent how the postulation of this impure set will commit us to a contradiction. Once we move from the naive context, of course, things look different. Applications of Replacement will give us paradoxical collections. Consider, for example a function that is onto from our *urelemente* set to the non self-membered sets. Replacement gives us that the image of our set under this function is itself a set, and Russell's paradox is immediate<sup>43</sup>. However, the context of our present discussion is logically *prior* to the axiomatisation of set theory. We are concerned with motivating principles. Why should we believe Replacement, and the limitation of size principle (of an LOS-sort) it is naturally understood as embodying? Why shouldn't our *urelemente* set exist?

None of this is intended to suggest that Replacement misleads set theorists in practice. So long as Replacement doesn't deliver false conclusions with respect to the sets whose existence can be proved from ZF, or its strengthenings in terms of customary large cardinal axioms etc., we can acquiesce, and there is no reason to believe that it does. This comfort does not help, though, when we are concerned with giving an explanatory answer to the question of set formation. If there is doubt about one direction of (LOS), then we do not have an answer to that question. We can more confidently affirm:

**(WEAK-LOS)** A plurality forms a set if it has fewer members than there are ordinals.

But this is only a partial answer.

1.3.3. *Set theoretic naturalism.* We have an answer to the question of set formation which is arguably true but unexplanatory, (IT-SET); we have another answer which, whilst often invoked, we are not justified in believing (LOS); and we have a partial answer, (WEAK-LOS). Where does this

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<sup>42</sup>So, in particular, consider it apart from antecedent acceptance of Replacement, in whatever form. Of course, it is the case that with Replacement in our set theory, a contradiction is quickly forthcoming here if the *urelemente* are onto the ordinals. But we are considering the motivation of Limitation of Size, as a candidate answer to the question of set formation. Replacement is too closely related to Limitation of Size for the latter to be considered when the former has already been accepted, so Replacement must be discounted for present dialectical purposes.

<sup>43</sup>For a suggested fix see [Menzel, 2014].

leave occasionalism? To approach this from a different angle, think about an epistemological analogue to the question of set formation:

(SBQ) When are we justified in believing that a given plurality forms a set?

This invites a response we might term *naturalistic*<sup>44</sup>:

(NAT) We are justified in believing that a plurality forms a set iff our best set theory confirms that it does.

This is deeply plausible. What extra set-theoretic reasons could there be to believe that a set exists? Set theory just is the study of the sets. Now, whilst (NAT) answers an *epistemological* question distinct from the *metaphysical* question of set formation, it does contain the seed of a deflationary approach to the latter. Under what conditions does a plurality form a set? Don't expect an answer that is both general and explanatory: look to your set theory, and believe (defeasibly) that those all and only pluralities form sets which your theory tells you do. To the extent that your theory is still being constructed, the subject of debate, or open-ended, so too is your capacity to answer the formation question in a deep fashion – there is, of course, a fact of the matter which pluralities form sets; it is simply hidden from you. Does this mean philosophers can say nothing at all in response to the question of set formation? (IT-SET) is true, and tells us something about the structure of the set theoretic universe, which can itself be the object of philosophical reflection. (NAT) is a partial answer to the set theoretic question which captures something of the reasoning motivating the acceptance of a set theoretic axiom, and might itself be reflected upon fruitfully. In general, philosophers can analyse the methods and motivating reasons of set theorists, as they can those of other scientists. They can also address questions about the metaphysics of the sets themselves.

This deflationary conclusion might seem poor compared to the hopes we may have had when we first asked the set theoretic question. However a naturalistic occasionalism answer is far from philosophical quietism. As an occasionalism, it emphasises the need to disambiguate the concepts *set* and *plurality* and it does not rule out philosophically substantial conceptions of sets playing a part in determining which partially explanatory answers to the question there may be. What pessimism there remains relates to the possibility of an across the board formula for saying *of any given plurality* whether it will form a set. But is the set question here in so very different a situation from the Special Composition Question, by means of comparison with which our discussion began? Suppose I am a mereological occasionalist, and more specifically claim that all and only those pluralities form

<sup>44</sup>On naturalistic approaches to the philosophy of mathematics see [Maddy, 1997].

fusions which my best scientific theory tells me do, *i.e.* that I believe in all and only those composite objects which my science commits me to. Now, I have necessary and sufficient conditions for fusion, and I can shed some philosophical light on the conditions under which a scientific theory might commit me to fusions: I may have a general theory of scientific explanations, I might be able to unfold a case for particular composite objects – living organisms, say – on the basis of considerations from the philosophy of a specific science. But can I say, in advance of scientific investigation, which fusions there are? Of course not, and it is not to the detriment of my metaphysics that it does not deliver an answer to this question. As with fusions, so with sets. As a philosopher, I cannot answer the question which sets there are: that is a task for the mathematician<sup>45</sup>. I can however propose general, albeit relatively non-explanatory, necessary and sufficient conditions, and discuss questions around set formation<sup>46</sup> This is philosophy's proper task in the area, it can give rise to fruitful discussions, and it is one with which philosophers should be content.<sup>47</sup>

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<sup>45</sup>Conceding this in no way involves a denial that a working set theorist might invoke reasons that look *philosophical* in advance of her position. Indeed, this clearly happens frequently at the research frontier (here, perhaps, set theory is no different from foundational parts of theoretical physics). What is ruled out is philosophical dictation of set theoretic possibility prior to and apart from the mathematical enterprise and the suggestion that there is some distinctively philosophical insight that can resolve substantial set theoretic questions – by supplying a 'recipe' for inferring the existence of sets from that of pluralities, for example. My attitude towards mathematics here is similar to that towards science in general outlined by Maddy in [Maddy, 2007].

<sup>46</sup>Note that the plural logic **PFO**-, that is impredicative **PFO** without nonlogical predicates in the language and the corresponding proof theoretic resources is easily provided by induction on complexity of formulae to be bi-interpretable with monadic second-order logic. Thus, we can understand plural set theory as interpreting ZF2 or, admitting nonstandard models of the plural logic [Linnebo, 2011], Morse-Kelley class theory. This provides some answer to a referee's concern that plural quantification plays little role in set theory, and thus that the approach of this paper is in tension with its naturalism. We can understand *class* quantification as plural – this idea is already mooted by Boolos and Uzquiano. This suggests another sort of principle which might be thought to provide substantive answers to the question which pluralities form a set - namely *reflection* principles. I expect the dialectic to play out in this case as with the other principles mooted, but further investigation is merited.

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